Math 210

Quiz # 2, April 26, 2010

1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. State, without proof, four properties of f. Do not supply proofs, just give clear and precise statements of those properties.

2. Let $f : [a, b] \to \mathbb{R}$ and $x \in [a, b]$. If f is continuous at the point x, and $f(x) \neq 0$, prove that $\frac{1}{f}$ is continuous at x. Give an $\epsilon - \delta$ proof. If you need a property of f related to those properties in problem 1, state that property clearly, but do not prove it.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \left\{ \begin{array}{cc} x^2 \sin \frac{5}{x} & x \neq 0 \\ 0 & x = 0 \end{array} \right\}.$$

(a) Prove that f is differentiable everywhere on $\mathbb{R}$ and compute $f'(x)$ for each $x \in \mathbb{R}$.

(b) Prove that f' is bounded on $\mathbb{R}$. (c) Prove that f' is not continuous at 0.

4. Prove the inequality

$$e^x < \frac{2+x}{2-x}\ 0 < x < 2.$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, and suppose that there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t. Let $x_0 \in \mathbb{R}$, and define a sequence $\{x_n\}$ by

$$x_{n+1} = \frac{2x_n + 3f(x_n)}{5}, n = 0, 1, 2, \cdots$$

Prove that the sequence $\{x_n\}$ is convergent, and that its limit is the unique fixed point of f.

6. Suppose (a) f is continuous on $[0, \infty)$; (b) differentiable on $(0, \infty)$; (c) $f(0) = 0$; (d) f' is monotonically increasing. Put $g(x) = \frac{f(x) - 3\sqrt{x}}{x}$, $x > 0$, and prove that g is monotonically increasing.